A GLOBAL PINCHING THEOREM OF COMPLETE λ-HYPERSURFACES

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ABSTRACT. In this paper, the pinching problems of complete λ-hypersurfaces in a Euclidean space $\mathbb{R}^{n+1}$ are studied. By making use of the Sobolev inequality, we prove a global pinching theorem of complete λ-hypersurfaces in a Euclidean space $\mathbb{R}^{n+1}$.

1. Introduction

Let M^n be an n-dimensional manifold, and $X : M^n \to \mathbb{R}^{n+1}$ an immersed hypersurface in a Euclidean space $\mathbb{R}^{n+1}$. If $X : M^n \to \mathbb{R}^{n+1}$ satisfies

$$H + \langle X,\ N\rangle = 0,$$

one calls that $X : M^n \to \mathbb{R}^{n+1}$ is a self-shrinker of mean curvature flow, where H and N are the mean curvature and the unit normal vector of $X : M^n \to \mathbb{R}^{n+1}$, respectively, and $\langle \cdot,\ \cdot\rangle$ is the standard inner product of $\mathbb{R}^{n+1}$.

Remark 1.1. *If $X : M^n \to \mathbb{R}^{n+1}$ is a self-shrinker of mean curvature flow, then $X(t) = \sqrt{1-2t}X$ is a self-similar solution of mean curvature flow.*

It is well-known that the n-dimensional Euclidean space $\mathbb{R}^n$, the n-dimensional sphere $S^n(\sqrt{n})$ and the n-dimensional cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, for $1 \le k \le n-1$, are the standard self-shrinkers of mean curvature flow. For the other examples of self-shrinkers of mean curvature flow, see [1], [8], [9], [10] and [13].

$X(t) : M^n \to \mathbb{R}^{n+1}$ is called a variation of $X : M^n \to \mathbb{R}^{n+1}$ if $X(t) : M^n \to \mathbb{R}^{n+1}, t \in (-\varepsilon, \varepsilon)$, are a family of immersions with $X(0) = X$. We define a weighted area functional $A : (-\varepsilon, \varepsilon) \to \mathbb{R}$ as follows:

$$A(t) = \int_M e^{-\frac{|X(t)|^2}{2}} d\mu_t,$$

where $d\mu_t$ is the area element of $X(t) : M^n \to \mathbb{R}^{n+1}$. In [6], Colding and Minicozzi have proved that that $X : M^n \to \mathbb{R}^{n+1}$ is a critical point of the weighted area functional $A(t)$ if and only if $X : M^n \to \mathbb{R}^{n+1}$ is a self-shrinker of mean curvature flow.

In [2], Cao and Li (cf. [6], [4] [11] and [14]) have proved a gap theorem of complete self-shrinkers of mean curvature flow as follows:

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Theorem 1.1. *Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional complete proper self-shrinker in $\mathbb{R}^{n+1}$. If the squared norm S of the second fundamental form of $X : M^n \to \mathbb{R}^{n+1}$ satisfies $S \leq 1$, then $X : M^n \to \mathbb{R}^{n+1}$ is isometric to one of the following:*

1. *the sphere $S^n(\sqrt{n})$,*
2. *the Euclidean space $\mathbb{R}^n$,*
3. *the cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ for $1 \leq k \leq n-1$.*

By using the following Sobolev inequality for n-dimensional complete hypersurfaces:

$$\kappa^{-1}\left(\int_M g^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \leq \int_M |\nabla g|^2 d\mu + \frac{1}{2}\int_M H^2 g^2 d\mu, \ {}^\forall g \in C_c^\infty(M),$$

where $\kappa > 0$ is a constant, Ding and Xin [7] have proved a rigidity theorem of complete self-shrinkers of mean curvature flow as follows:

Theorem 1.2. *Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional complete immersed self-shrinker of mean curvature flow in $\mathbb{R}^{n+1}$. If $X : M^n \to \mathbb{R}^{n+1}$ satisfies*

$$\left(\int_M S^{\frac{n}{2}} d\mu\right)^{\frac{2}{n}} < \frac{4}{3n\kappa},$$

then $X : M^n \to \mathbb{R}^{n+1}$ is isometric to the Euclidean space $\mathbb{R}^n$, where S denotes the squared norm of the second fundamental form of $X : M^n \to \mathbb{R}^{n+1}$.

In [5], Cheng and Wei have introduced a notation of so-called λ-hypersurfaces of the weighted volume-preserving mean curvature as follows:

Definition 1.1. *Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional immersed hypersurface in $\mathbb{R}^{n+1}$. If*

$$H + \langle X,\ N\rangle = \lambda$$

is satisfied, where λ is constant, then $X : M^n \to \mathbb{R}^{n+1}$ is called a λ-hypersurface of the weighted volume-preserving mean curvature. For simple, we call it a λ-hypersurface.

Remark 1.2. *From definition, we know that if $\lambda = 0$, $X : M^n \to \mathbb{R}^{n+1}$ is a self-shrinker of mean curvature flow.*

Example 1.1. *All of self-shrinkers of mean curvature flow is λ-hypersurfaces with $\lambda = 0$.*

Example 1.2. *The n-dimensional sphere $S^n(r)$ with $r > 0$ is a compact λ-hypersurface with $\lambda = \frac{n}{r} - r$. We should notice that only the n-dimensional sphere $S^n(\sqrt{n})$ is the self-shrinker of mean curvature flow.*

Example 1.3. *The n-dimensional cylinder $S^k(r)\times\mathbb{R}^{n-k}$ with $r > 0$ for $1 \leq k \leq n-1$ is a complete and non-compact λ-hypersurface with $\lambda = \frac{k}{r} - r$. We remark that only the n-dimensional cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, for $1 \leq k \leq n-1$, is the self-shrinker of mean curvature flow.*

Let $X(t) : M^n \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ be a variation of $X : M^n \to \mathbb{R}^{n+1}$. The weighted volume $V : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is defined in [5] as follows:

$$V(t) = \int_M \langle X(t),\ N\rangle\, e^{-\frac{|X|}{2}} d\mu.$$

In [5], Cheng and Wei have proved that $X : M^n \to \mathbb{R}^{n+1}$ is a critical point of the weighted area functional $A(t)$ for the weighted volume-preserving variations if and only if $X : M^n \to \mathbb{R}^{n+1}$ is a λ-hypersurface. For further properties of λ-hypersurfaces in details, see [5].

In [3], Cheng, Ogata and Wei have studied the rigidity theorem of complete λ-hypersurfaces with a pointwise pinching condition. The following theorem is proved:

Theorem 1.3. *Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional complete proper λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If the squared norm S of the second fundamental form and the mean curvature H of $X : M^n \to \mathbb{R}^{n+1}$ satisfies*

$$\left(\sqrt{S - \frac{H^2}{n}} + |\lambda| \frac{n-2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H-\lambda)^2 \le 1 + \frac{n\lambda^2}{4(n-1)}, \tag{1.1}$$

then $X : M^n \to \mathbb{R}^{n+1}$ is isometric to one of the following:

1. *the sphere $S^n(r)$ with radius $0 < r \le \sqrt{n}$,*
2. *the Euclidean space $\mathbb{R}^n$,*
3. *the cylinder $S^1(r) \times \mathbb{R}^{n-1}$ with radius $r > 0$ and $n = 2$ or with radius $r \ge 1$ and $n > 2$,*
4. *the cylinder $S^{n-1}(r) \times \mathbb{R}$ with radius $r > 0$ and $n = 2$ or with radius $r \le \sqrt{n-1}$ and $n > 2$,*
5. *the cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ for $2 \le k \le n-2$.*

In this paper, we study a global pinching theorem of complete λ-hypersurfaces in $\mathbb{R}^{n+1}$. We prove the following:

Theorem 1.4. *Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional complete proper λ-hypersurface in $\mathbb{R}^{n+1}$ with $n \ge 3$. If $X : M^n \to \mathbb{R}^{n+1}$ satisfies*

$$\begin{aligned} &\left(\int_M \left| \frac{n|\lambda|(n-2)}{2\sqrt{n(n-1)}} B^{\frac{1}{2}} + \frac{n}{2} B + \frac{n^2-2n+2}{2n(n-1)} H^2 - \frac{n+2}{2n}\lambda H \right|^{\frac{n}{2}} d\mu \right)^{\frac{2}{n}} \\ &< \frac{n-2}{n} k(n)^{-1}, \end{aligned} \tag{1.2}$$

then $X : M^n \to \mathbb{R}^{n+1}$ is isometric to the Euclidean space $\mathbb{R}^n$ or the sphere $S^n(r)$ with

$$\left| r^2 - \frac{(3n-4)n}{(n-1)(n+2)} \right| < \frac{(n-2)^3}{4^{2(n+1)} n^{\frac{2}{n}+1} (n-1)(3n-4)(n+2)} \left(\frac{\omega_{n-1}}{\omega_n}\right)^{\frac{2}{n}}, \tag{1.3}$$

where $B = S - \dfrac{H^2}{n}$, $k(n) = \frac{2\cdot 4^{2(n+1)}(n-1)(3n-4)}{(n-2)^2} \left(\frac{n}{\omega_{n-1}}\right)^{\frac{2}{n}}$, and ω_k denotes the area of the k-dimensional unit sphere $S^k(1)$.

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2. the Sobolev inequality

In order to prove our theorem, the following Sobolev inequality in [12] plays a very important rule.

Theorem 2.1. *Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional hypersurface in $\mathbb{R}^{n+1}$. For any Lipschitz function $f \geq 0$ with compact support on M,*

$$\left(\int_M f^{\frac{n}{n-1}} d\mu\right)^{\frac{n-1}{n}} \leq C_n \int_M \{|\nabla f| + |H| f\} \, d\mu \tag{2.1}$$

holds, where

$$C_n = 4^{n+1} \left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{n}}.$$

From the above theorem, we have the following corollary:

Corollary 2.1. *Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional hypersurface in $\mathbb{R}^{n+1}$. For any Lipschitz function $f \geq 0$ with compact support on M,*

$$k(n)^{-1} \left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \leq \int_M |\nabla f|^2 d\mu + \frac{n-2}{2(n-1)} \int_M H^2 f^2 d\mu \tag{2.2}$$

holds, where

$$k(n) = \frac{2 \cdot 4^{2(n+1)} (n-1)(3n-4)}{(n-2)^2} \left(\frac{n}{\omega_{n-1}}\right)^{\frac{2}{n}}.$$

Proof. Replacing f in the theorem 2.1 with $f^{\frac{2(n-1)}{n-2}}$, we get

$$\begin{aligned}
\left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-1}{n}} &\leq C_n \int_M \left\{|\nabla f^{\frac{2(n-1)}{n-2}}| + |H| f^{\frac{2(n-1)}{n-2}}\right\} d\mu \\
&= C_n \int_M \left\{\frac{2(n-1)}{n-2} f^{\frac{n}{n-2}} |\nabla f| + |H| f^{\frac{2(n-1)}{n-2}}\right\} d\mu.
\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
\int_M f^{\frac{n}{n-2}} |\nabla f| d\mu &\leq \left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{1}{2}} \left(\int_M |\nabla f|^2 d\mu\right)^{\frac{1}{2}}, \\
\int_M |H| f^{\frac{2(n-1)}{n-2}} d\mu &\leq \left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{1}{2}} \left(\int_M H^2 f^2 d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-1}{n}} &\leq C_n \frac{2(n-1)}{n-2} \left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{1}{2}} \left(\int_M |\nabla f|^2 d\mu\right)^{\frac{1}{2}} \\
&\quad + C_n \left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{1}{2}} \left(\int_M H^2 f^2 d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore

$$\left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{2n}} \leq C_n \left\{ \frac{2(n-1)}{n-2} \left(\int_M |\nabla f|^2 d\mu\right)^{\frac{1}{2}} + \left(\int_M H^2 f^2 d\mu\right)^{\frac{1}{2}} \right\}.$$

According to $\left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{2n}} \geq 0$, we have

$$\begin{aligned}\left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \leq\ & {C_n}^2 \left\{ \frac{4(n-1)^2}{(n-2)^2} \int_M |\nabla f|^2 d\mu + \int_M H^2 f^2 d\mu \right. \\ & \left. + \frac{4(n-1)}{n-2} \left(\int_M |\nabla f|^2 d\mu\right)^{\frac{1}{2}} \left(\int_M H^2 f^2 d\mu\right)^{\frac{1}{2}} \right\}.\end{aligned}$$

Because of $\int_M |\nabla f|^2 d\mu \geq 0$ and $\int_M H^2 f^2 d\mu \geq 0$, we get

$$\begin{aligned} & \left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \\ \leq\ & {C_n}^2 \left\{ \frac{4(n-1)^2}{(n-2)^2} \int_M |\nabla f|^2 d\mu + \int_M H^2 f^2 d\mu \right. \\ & \left. + \frac{2(n-1)}{n-2} \left(\int_M |\nabla f|^2 d\mu + \int_M H^2 f^2 d\mu \right) \right\} \\ =\ & {C_n}^2 \frac{2(n-1)}{n-2} \left(\frac{2(n-1)}{n-2} + 1 \right) \left\{ \int_M |\nabla f|^2 d\mu + \frac{n-2}{2(n-1)} \int_M H^2 f^2 d\mu \right\}. \end{aligned}$$

Let $k(n) = {C_n}^2 \frac{2(n-1)}{n-2} \left(\frac{2(n-1)}{n-2} + 1 \right)$, then we get

$$k(n)^{-1} \left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \leq \int_M |\nabla f|^2 d\mu + \frac{n-2}{2(n-1)} \int_M H^2 f^2 d\mu.$$

□

3. Proof of our global pinching theorem

In order to prove the theorem 1.4, we prepare several lemmas. For the differential operator $\mathcal{L}$ defined by

$$\mathcal{L} f = \Delta f - \langle \nabla f, X \rangle = div(e^{-\frac{|X|^2}{2}} \nabla f) e^{\frac{|X|^2}{2}},$$

where Δ and ∇ denote the Laplace operator and the gradient operator. In [3], we have proved the following lemma.

Lemma 3.1. *For $B = S - \frac{H^2}{n}$, we have*

$$\frac{1}{2} \mathcal{L} B \geq - \frac{|\lambda|(n-2)}{\sqrt{n(n-1)}} B^{\frac{3}{2}} + B - B^2 - \frac{1}{n} H^2 B + \frac{2\lambda}{n} H B. \tag{3.1}$$

Define a function ρ by

$$\rho = e^{-\frac{|X|^2}{2}}.$$

Lemma 3.2. *For any smooth function η with compact support on M and an arbitrary positive constant ε, we have*

$$\begin{aligned}
(3.2) \qquad & \int_M \left\{ \frac{|\lambda|(n-2)}{\sqrt{n(n-1)}} B^{n+\frac{1}{2}}\eta^2\rho - B^n\eta^2\rho + B^{n+1}\eta^2\rho \right. \\
& \left. + \frac{1}{n}H^2B^n\eta^2\rho - \frac{2\lambda}{n}HB^n\eta^2\rho + \frac{1}{2\varepsilon}B^n|\nabla\eta|^2\rho \right\} d\mu \\
\geq \ & \frac{n-1-\varepsilon}{2}\int_M B^{n-2}\eta^2|\nabla B|^2\rho \, d\mu.
\end{aligned}$$

Proof. Multiplying $B^{n-1}\eta^2\rho$ on both sides of (3.1) and taking integral, we obtain

$$\begin{aligned}
0 \ \geq \ & \int_M \left\{ -\frac{|\lambda|(n-2)}{\sqrt{n(n-1)}} B^{n+\frac{1}{2}}\eta^2\rho + B^n\eta^2\rho - B^{n+1}\eta^2\rho \right. \\
& \left. -\frac{1}{n}H^2B^n\eta^2\rho + \frac{2\lambda}{n}HB^n\eta^2\rho - \frac{1}{2}\mathcal{L}B\cdot B^{n-1}\eta^2\rho \right\} d\mu.
\end{aligned}$$

Since η has compact support on M, according to Stokes theorem, we get

$$\begin{aligned}
& -\frac{1}{2}\int_M \mathcal{L}B\cdot B^{n-1}\eta^2\rho \, d\mu \\
= \ & -\frac{1}{2}\int_M div(\rho\nabla B)\cdot B^{n-1}\eta^2 d\mu \\
= \ & \frac{1}{2}\int_M \left\langle \rho\nabla B, \ \nabla(B^{n-1}\eta^2)\right\rangle d\mu \\
= \ & \frac{n-1}{2}\int_M B^{n-2}\eta^2|\nabla B|^2\rho \, d\mu + \int_M B^{n-1}\eta \left\langle \nabla B, \ \nabla\eta\right\rangle \rho \, d\mu.
\end{aligned}$$

Moreover, for an arbitrary constant $\varepsilon > 0$, we have

$$\int_M B^{n-1}\eta\left\langle \nabla B, \ \nabla\eta\right\rangle\rho \, d\mu \geq -\frac{\varepsilon}{2}\int_M B^{n-2}\eta^2|\nabla B|^2\rho \, d\mu - \frac{1}{2\varepsilon}\int_M B^n|\nabla\eta|^2\rho \, d\mu.$$

Hence, we obtain

$$\begin{aligned}
& \int_M \left\{ \frac{|\lambda|(n-2)}{\sqrt{n(n-1)}} B^{n+\frac{1}{2}}\eta^2\rho - B^n\eta^2\rho + B^{n+1}\eta^2\rho \right. \\
& \left. + \frac{1}{n}H^2B^n\eta^2\rho - \frac{2\lambda}{n}HB^n\eta^2\rho + \frac{1}{2\varepsilon}B^n|\nabla\eta|^2\rho \right\} d\mu \\
\geq \ & \frac{n-1-\varepsilon}{2}\int_M B^{n-2}\eta^2|\nabla B|^2\rho \, d\mu.
\end{aligned}$$

$\square$

Lemma 3.3. *Putting* $f := B^{\frac{n}{2}}\eta\rho^{\frac{1}{2}}$, *we know that*

$$\begin{aligned}(3.3)\quad & \int_M |\nabla f|^2 d\mu \\ = \; & \frac{n^2}{4}\int_M B^{n-2}\eta^2|\nabla B|^2\rho\, d\mu + \int_M B^n|\nabla\eta|^2\rho\, d\mu + n\int_M B^{n-1}\eta\left<\nabla B,\ \nabla\eta\right>\rho\, d\mu \\ & -\frac{1}{4}\int_M |X^\top|^2 B^n\eta^2\rho\, d\mu + \frac{\lambda}{2}\int_M \left<X,\ N\right> B^n\eta^2\rho\, d\mu - \frac{1}{2}|X^\perp|^2 B^n\eta^2\rho\, d\mu \\ & +\frac{n}{2}\int_M B^n\eta^2\rho\, d\mu\end{aligned}$$

and

$$\begin{aligned}(3.4)\quad & \frac{1}{2}\int_M H^2 f^2 d\mu \\ = \; & \frac{\lambda^2}{2}\int_M B^n\eta^2\rho\, d\mu - \lambda\int_M \left<X,\ N\right> B^n\eta^2\rho\, d\mu + \frac{1}{2}\int_M |X^\perp|^2 B^n\eta^2\rho\, d\mu.\end{aligned}$$

hold.

Proof. Calculating the left hand side of (3.3), we know

$$\begin{aligned}\int_M |\nabla f|^2 d\mu &= \int_M |\nabla(B^{\frac{n}{2}}\eta)|^2\rho\, d\mu + \int_M B^n\eta^2|\nabla\rho^{\frac{1}{2}}|^2 d\mu \\ & +2\int_M B^{\frac{n}{2}}\eta\rho^{\frac{1}{2}}\left<\nabla(B^{\frac{n}{2}}\eta),\ \nabla\rho^{\frac{1}{2}}\right> d\mu.\end{aligned}$$

Putting

$$\begin{aligned}T_1 &:= \int_M |\nabla(B^{\frac{n}{2}}\eta)|^2\rho\, d\mu, \\ T_2 &:= \int_M B^n\eta^2|\nabla\rho^{\frac{1}{2}}|^2 d\mu, \\ T_3 &:= 2\int_M B^{\frac{n}{2}}\eta\rho^{\frac{1}{2}}\left<\nabla(B^{\frac{n}{2}}\eta),\ \nabla\rho^{\frac{1}{2}}\right> d\mu.\end{aligned}$$

$$\begin{aligned}T_1 &= \frac{n^2}{4}\int_M B^{n-2}\eta^2|\nabla B|^2\rho\, d\mu + \int_M B^n|\nabla\eta|^2\rho\, d\mu \\ & +n\int_M B^{n-1}\eta\left<\nabla B,\ \nabla\eta\right>\rho\, d\mu.\end{aligned}$$

Because of $|\nabla\rho^{\frac{1}{2}}|^2 = \frac{1}{4}|X^\top|^2\rho$ and $\Delta X = HN$, we have

$$T_2 = \frac{1}{4}\int_M |X^\top|^2 B^n\eta^2\rho\, d\mu$$

and

$$\begin{aligned}\Delta\rho &= |X^\top|^2\rho - \left<\Delta X,\ X\right>\rho - n\rho \\ &= |X^\top|^2\rho - \lambda\left<X,\ N\right>\rho + |X^\perp|^2\rho - n\rho.\end{aligned}$$

Hence,

$$\begin{aligned} T_3 &= \frac{1}{2}\int_M \left\langle \nabla(B^n\eta^2),\ \nabla\rho \right\rangle d\mu \\ &= -\frac{1}{2}\int_M B^n\eta^2\cdot\Delta\rho\ d\mu \\ &= -\frac{1}{2}\int_M |X^\top|^2 B^n\eta^2\rho\ d\mu + \frac{\lambda}{2}\int_M \langle X,\ N\rangle B^n\eta^2\rho\ d\mu \\ &\quad -\frac{1}{2}\int_M |X^\perp|^2 B^n\eta^2\rho\ d\mu + \frac{n}{2}\int_M B^n\eta^2\rho\ d\mu. \end{aligned}$$

Therefore, we get

$$\begin{aligned} &\int_M |\nabla f|^2 d\mu \\ = \ &\frac{n^2}{4}\int_M B^{n-2}\eta^2|\nabla B|^2\rho\ d\mu + \int_M B^n|\nabla\eta|^2\rho\ d\mu + n\int_M B^{n-1}\eta\left\langle \nabla B,\ \nabla\eta\right\rangle\rho\ d\mu \\ &-\frac{1}{4}\int_M |X^\top|^2 B^n\eta^2\rho\ d\mu + \frac{\lambda}{2}\int_M \langle X,\ N\rangle B^n\eta^2\rho\ d\mu - \frac{1}{2}|X^\perp|^2 B^n\eta^2\rho\ d\mu \\ &+\frac{n}{2}\int_M B^n\eta^2\rho\ d\mu. \end{aligned}$$

From $H = \lambda - \langle X,\ N\rangle$, we get

$$\begin{aligned} \frac{1}{2}\int_M H^2 f^2 d\mu &= \frac{1}{2}\int_M (\lambda - \langle X,\ N\rangle)^2 B^n\eta^2\rho\ d\mu \\ &= \frac{\lambda^2}{2}\int_M B^n\eta^2\rho\ d\mu - \lambda\int_M \langle X,\ N\rangle B^n\eta^2\rho\ d\mu \\ &\quad +\frac{1}{2}\int_M |X^\perp|^2 B^n\eta^2\rho\ d\mu. \end{aligned}$$

□

Lemma 3.4. *For an arbitrary constant $\delta > 0$, we have*

$$\begin{aligned} &k(n)^{-1}\left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \qquad (3.5) \\ \leq\ &\frac{(1+\delta)n^2}{4}\int_M B^{n-2}\eta^2|\nabla B|^2\rho\ d\mu + \left(1+\frac{1}{\delta}\right)\int_M B^n|\nabla\eta|^2\rho\ d\mu \\ &-\frac{1}{2(n-1)}\int_M H^2 B^n\eta^2\rho\ d\mu + \frac{\lambda}{2}\int_M H B^n\eta^2\rho\ d\mu + \frac{n}{2}\int_M B^n\eta^2\rho\ d\mu, \end{aligned}$$

where $k(n)$ is the assertion of the Corollary 2.1.

Proof. From Corollary 2.1, we have, for any function f with compact support on M,

$$
\begin{aligned}
k(n)^{-1}\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-2}{n}} &\leq \int_M |\nabla f|^2 d\mu + \frac{n-2}{2(n-1)}\int_M H^2 f^2 d\mu \\
&= \int_M |\nabla f|^2 d\mu + \frac{1}{2}\int_M H^2 f^2 d\mu - \frac{1}{2(n-1)}\int_M H^2 f^2 d\mu.
\end{aligned}
$$

Taking $f = B^{\frac{n}{2}}\eta\rho^{\frac{1}{2}}$, from Lemma 3.3, we infer

$$
\begin{aligned}
& k(n)^{-1}\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-2}{n}} \\
\leq\ & \frac{n^2}{4}\int_M B^{n-2}\eta^2|\nabla B|^2\rho\, d\mu + \int_M B^n|\nabla\eta|^2\rho\, d\mu + n\int_M B^{n-1}\eta\left\langle \nabla B,\ \nabla\eta\right\rangle\rho\, d\mu \\
& -\frac{1}{4}\int_M |X^\top|^2 B^n\eta^2\rho\, d\mu + \frac{\lambda}{2}\int_M \left\langle X,\ N\right\rangle B^n\eta^2\rho\, d\mu - \frac{1}{2}|X^\perp|^2 B^n\eta^2\rho\, d\mu \\
& +\frac{n}{2}\int_M B^n\eta^2\rho\, d\mu + \frac{\lambda^2}{2}\int_M B^n\eta^2\rho\, d\mu - \lambda\int_M \left\langle X,\ N\right\rangle B^n\eta^2\rho\, d\mu \\
& +\frac{1}{2}\int_M |X^\perp|^2 B^n\eta^2\rho\, d\mu - \frac{1}{2(n-1)}\int_M H^2 B^n\eta^2\rho\, d\mu \\
\leq\ & \frac{n^2}{4}\int_M B^{n-2}\eta^2|\nabla B|^2\rho\, d\mu + \int_M B^n|\nabla\eta|^2\rho\, d\mu + n\int_M B^{n-1}\eta\left\langle \nabla B,\ \nabla\eta\right\rangle\rho\, d\mu \\
& -\frac{\lambda}{2}\int_M \left\langle X,\ N\right\rangle B^n\eta^2\rho\, d\mu - \frac{1}{2(n-1)}\int_M H^2 B^n\eta^2\rho\, d\mu + \left(\frac{n}{2}+\frac{\lambda^2}{2}\right)\int_M B^n\eta^2\rho\, d\mu \\
=\ & \frac{n^2}{4}\int_M B^{n-2}\eta^2|\nabla B|^2\rho\, d\mu + \int_M B^n|\nabla\eta|^2\rho\, d\mu + n\int_M B^{n-1}\eta\left\langle \nabla B,\ \nabla\eta\right\rangle\rho\, d\mu \\
& +\frac{\lambda}{2}\int_M HB^n\eta^2\rho\, d\mu - \frac{1}{2(n-1)}\int_M H^2 B^n\eta^2\rho\, d\mu + \frac{n}{2}\int_M B^n\eta^2\rho\, d\mu.
\end{aligned}
$$

For an arbitrary constant $\delta > 0$, we have

$$
n\int_M B^{n-1}\eta\left\langle \nabla B,\ \nabla\eta\right\rangle\rho\, d\mu \leq \frac{\delta n^2}{4}\int_M B^{n-2}\eta^2|\nabla B|^2\rho\, d\mu + \frac{1}{\delta}\int_M B^n|\nabla\eta|^2\rho\, d\mu.
$$

Hence, we get

$$
\begin{aligned}
& k(n)^{-1}\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-2}{n}} \\
\leq\ & \frac{(1+\delta)n^2}{4}\int_M B^{n-2}\eta^2|\nabla B|^2\rho\, d\mu + \left(1+\frac{1}{\delta}\right)\int_M B^n|\nabla\eta|^2\rho\, d\mu \\
& -\frac{1}{2(n-1)}\int_M H^2 B^n\eta^2\rho\, d\mu + \frac{\lambda}{2}\int_M HB^n\eta^2\rho\, d\mu + \frac{n}{2}\int_M B^n\eta^2\rho\, d\mu.
\end{aligned}
$$

$\square$

Proof of Theorem 1.4. If $B \not\equiv 0$ holds, we can choose η such that, for $f = B^{\frac{1}{2}}\eta\rho^{\frac{1}{2}}$,

$$\left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \neq 0.$$

From Lemma 3.2 and Lemma 3.4, then for arbitrary constants $\varepsilon > 0$ and $\delta > 0$,

$$\begin{aligned}
& k(n)^{-1}\left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \\
\leq\ & \frac{(1+\delta)n^2}{2}\frac{1}{n-1-\varepsilon}\int_M \left\{\frac{|\lambda|(n-2)}{\sqrt{n(n-1)}}B^{n+\frac{1}{2}}\eta^2\rho - B^n\eta^2\rho + B^{n+1}\eta^2\rho\right. \\
& \left. + \frac{1}{n}H^2B^n\eta^2\rho - \frac{2\lambda}{n}HB^n\eta^2\rho + \frac{1}{2\varepsilon}B^n|\nabla\eta|^2\rho\right\} d\mu \\
& + \left(1+\frac{1}{\delta}\right)\int_M B^n|\nabla\eta|^2\rho\ d\mu - \frac{1}{2(n-1)}\int_M H^2B^n\eta^2\rho\ d\mu \\
& + \frac{\lambda}{2}\int_M HB^n\eta^2\rho\ d\mu + \frac{n}{2}\int_M B^n\eta^2\rho\ d\mu.
\end{aligned}$$

Letting $1+\delta = \dfrac{n-1+\varepsilon}{n}$, then, we derive

$$\begin{aligned}
& k(n)^{-1}\left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \\
\leq\ & \frac{n-1+\varepsilon}{n-1-\varepsilon}\int_M \left\{\frac{n|\lambda|(n-2)}{2\sqrt{n(n-1)}}B^{n+\frac{1}{2}}\eta^2\rho + \frac{n}{2}B^{n+1}\eta^2\rho\right. \\
& \left. + \frac{1}{2}\left(1 - \frac{1}{(n-1)}\frac{n-1-\varepsilon}{n-1+\varepsilon}\right)H^2B^n\eta^2\rho + \left(-1 + \frac{1}{2}\frac{n-1-\varepsilon}{n-1+\varepsilon}\right)\lambda HB^n\eta^2\rho\right\} d\mu \\
& + \frac{n}{2}\left(-\frac{n-1+\varepsilon}{n-1-\varepsilon} + 1\right)\int_M B^n\eta^2\rho\ d\mu + C(n,\varepsilon)\int_M B^n|\nabla\eta|^2\rho\ d\mu \\
\leq\ & \frac{n-1+\varepsilon}{n-1-\varepsilon}\int_M \left\{\frac{n|\lambda|(n-2)}{2\sqrt{n(n-1)}}B^{n+\frac{1}{2}}\eta^2\rho + \frac{n}{2}B^{n+1}\eta^2\rho\right. \\
& \left. + \frac{1}{2}\left(1 - \frac{1}{(n-1)}\frac{n-1-\varepsilon}{n-1+\varepsilon}\right)H^2B^n\eta^2\rho + \left(-1 + \frac{1}{2}\frac{n-1-\varepsilon}{n-1+\varepsilon}\right)\lambda HB^n\eta^2\rho\right\} d\mu \\
& + C(n,\varepsilon)\int_M B^n|\nabla\eta|^2\rho\ d\mu,
\end{aligned}$$

where $C(n,\varepsilon)$ is a positive constant only depending on n and ε. From $f^2 = B^n\eta^2\rho$ and using Hölder's inequality, we obtain

$$\begin{aligned}
& k(n)^{-1}\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-2}{n}} \\
\leq\ & \frac{n-1+\varepsilon}{n-1-\varepsilon}\Bigg(\int_M\Bigg|\frac{n|\lambda|(n-2)}{2\sqrt{n(n-1)}}B^{\frac{1}{2}}+\frac{n}{2}B+\frac{1}{2}\left(1-\frac{1}{(n-1)}\frac{n-1-\varepsilon}{n-1+\varepsilon}\right)H^2 \\
& +\left(-1+\frac{1}{2}\frac{n-1-\varepsilon}{n-1+\varepsilon}\right)\lambda H\Bigg|^{\frac{n}{2}}d\mu\Bigg)^{\frac{2}{n}}\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-2}{n}} \\
& +C(n,\varepsilon)\int_M B^n|\nabla\eta|^2\rho\ d\mu.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& k(n)^{-1} \\
\leq\ & \frac{n-1+\varepsilon}{n-1-\varepsilon}\Bigg(\int_M\Bigg|\frac{n|\lambda|(n-2)}{2\sqrt{n(n-1)}}B^{\frac{1}{2}}+\frac{n}{2}B+\frac{1}{2}\left(1-\frac{1}{(n-1)}\frac{n-1-\varepsilon}{n-1+\varepsilon}\right)H^2 \\
& +\left(-1+\frac{1}{2}\frac{n-1-\varepsilon}{n-1+\varepsilon}\right)\lambda H\Bigg|^{\frac{n}{2}}d\mu\Bigg)^{\frac{2}{n}} \\
& +C(n,\varepsilon)\frac{\int_M B^n|\nabla\eta|^2\rho\ d\mu}{\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-2}{n}}}.
\end{aligned}$$

Since $X: M^n\to\mathbb{R}^{n+1}$ is proper, it is proved by Cheng and Wei in [5] that $X: M^n\to\mathbb{R}^{n+1}$ has at most polynomial area growth. Hence, we know that

$$\int_M B^n\rho\ d\mu<\infty.$$

Taking $\eta=\phi(\frac{|X|}{r})$ for any $r>0$, where ϕ is a nonnegative function on $[0,\infty)$ such that

$$\phi(t)=\begin{cases}1, & \text{if } t\in[0,1]\\ 0, & \text{if } t\in[2,\infty)\end{cases}$$

and $|\phi'|\leq c$ for some absolute constant. Taking $r\to\infty$, we have

$$\int_M B^n|\nabla\eta|^2\rho\ d\mu\to 0.$$

Therefore, we get

$$\begin{aligned}
&k(n)^{-1}\\
\leq\ &\frac{n-1+\varepsilon}{n-1-\varepsilon}\left(\int_M\left|\frac{n|\lambda|(n-2)}{2\sqrt{n(n-1)}}B^{\frac{1}{2}}+\frac{n}{2}B+\frac{1}{2}\left(1-\frac{1}{(n-1)}\frac{n-1-\varepsilon}{n-1+\varepsilon}\right)H^2\right.\right.\\
&\left.\left.+\left(-1+\frac{1}{2}\frac{n-1-\varepsilon}{n-1+\varepsilon}\right)\lambda H\right|^{\frac{n}{2}}d\mu\right)^{\frac{2}{n}}.
\end{aligned}$$

Letting $\varepsilon\to 1$, we obtain

$$\begin{aligned}
&k(n)^{-1}\\
&\leq\frac{n}{n-2}\left(\int_M\left|\frac{n|\lambda|(n-2)}{2\sqrt{n(n-1)}}B^{\frac{1}{2}}+\frac{n}{2}B+\frac{n^2-2n+2}{2n(n-1)}H^2-\frac{n+2}{2n}\lambda H\right|^{\frac{n}{2}}d\mu\right)^{\frac{2}{n}}\\
&<\frac{n}{n-2}\cdot\frac{n-2}{n}k(n)^{-1}=k(n)^{-1}.
\end{aligned}$$

It is a contradiction. Thus, we have $B=S-\frac{H^2}{n}\equiv 0$, that is, $X:M^n\to\mathbb{R}^{n+1}$ is totally umbilical. Hence, we know that $X:M^n\to\mathbb{R}^{n+1}$ is isomeric to $\mathbb{R}^n$ or a sphere $S^n(r)$ with radius r, which satisfies (1.3) from (1.2).

$\square$

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